

Fixed point theorems in complete 2-metric spaces by using A Continuous control function

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Abstract

The main purpose of this paper is to obtain fixed points for self maps on a complete 2-metric spaces under a more general contraction type condition by using a certain continuous control function. Further generalization of this theorem for a pair of self maps is given, when the complete 2-metric space is bounded.

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Introduction

Park (1980) and Khan, Swaleh and S.Sess (1984) established a new technique to obtain fixed points for self maps on a 2-metric space by altering distances between the points with the use of a certain continuous control function. Pzthak and Sharma (1994), Sastry and Babu (1998, 1999) worked in this direction. In fact, Sastry and Babu (1999) discussed and established the existence of fixed points for the orbits of single self maps

and pairs of self maps by using a control function. The purpose of using a control function is that it verifies and generalizes many known results.

Throughout this paper, \mathbb{R}^+ denotes the set of all non-negative real numbers \mathbb{N} , the set of all natural numbers and Φ the set of all continuous self maps φ of \mathbb{R}^+ satisfying (i) φ is monotonically increasing and (ii) $\varphi(\epsilon) = 0$ iff $\epsilon = 0$.

Fixed Point Theorem for a single self map Theorem 1.

Let (X, ρ) be a complete 2-metric space. T a self map of X ($T: X \rightarrow X$). Assume that T satisfies the following inequality:

$$\varphi(\rho(Tx, Ty, z)) \leq K \max\{\varphi(\rho(x, y, z)), \varphi(\rho(x, Tx, z)), \varphi(\rho(y, Ty, z))\} \quad (1)$$

for all $x, y, z \in X$, $K \in (0, 1)$, $\varphi \in \Phi$. Then T has a unique fixed point in X .

Proof. Let $x_0 \in X$. Then define the sequence $\{x_n\}$ in X by $x_n = T^n x_0$ for $n = 1, 2, 3, \dots$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then $Tx_n = x_{n+1} = x_n$ such that x_n

is a fixed point for T. Suppose $x_n \neq x_{n+1}$ for some $n \in \mathbb{N}$. Take $\beta_n = \rho(x_n, x_{n+1}, z)$ and $\alpha_n = \varphi(\beta_n)$ for $n = 0, 1, 2, \dots, (z \in X)$.

$$\begin{aligned} \text{By (1), } \alpha_1 &= \varphi(\beta_1) = \varphi(\rho(x_1, x_2, z)) = \varphi(\rho(Tx_0, Tx_1, z)) \\ &\leq K \max\{ \varphi(\rho(x_0, x_1, z)), \varphi(\rho(x_0, Tx_0, z)), \varphi(\rho(x_1, Tx_1, z)) \} \\ &= K \varphi(\rho(x_0, Tx_0, z)) \end{aligned}$$

$$\Rightarrow \alpha_1 \leq K\alpha_0.$$

In general, we can show that $\alpha_n \leq K\alpha_{n-1}$. (2)

By induction method, it is easy to see that $\alpha_n \leq K^n \alpha_0$.

Since $K^n \rightarrow 0$ as $n \rightarrow \infty$, $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$. (3)

By (2), $\alpha_n \leq \alpha_{n-1}$ implies $\beta_n \leq \beta_{n-1}$ for $n = 1, 2, 3, \dots$.

Hence $\{\beta_n\}$ is a decreasing sequence of non-negative numbers.

Let $\{\beta_n\} \rightarrow \beta$ as $n \rightarrow \infty$. Then, by the continuity of φ ,

$$\alpha_n = \varphi(\beta_n) \rightarrow \varphi(\beta) \text{ as } n \rightarrow \infty.$$

Hence from (3), it follows that $\varphi(\beta) = 0$, which shows that $\beta = 0$.

$$\text{i.e., } \lim_{n \rightarrow \infty} \rho(x_n, x_{n+1}, z) = 0 \tag{4}$$

We now prove that $\{x_n\}$ is a Cauchy sequence. Otherwise, there exists an $\varepsilon > 0$ and a sequence $\{m(p)\}$ and $n(p)$ such that $m(p) < n(p)$

with

$$\rho(x_{m(p)}, x_{n(p)}, z) \geq \varepsilon \text{ and } \rho(x_{m(p)}, x_{n(p)-1}, z) < \varepsilon.$$

3Hence

$$\begin{aligned} \varphi(\varepsilon) &\leq \varphi(\rho(x_{m(p)}, x_{n(p)}, z)) \\ &= \varphi(\rho(Tx_{m(p)-1}, Tx_{n(p)-1}, z)) \\ &\leq K \max\{ \varphi(\rho(x_{m(p)-1}, x_{n(p)-1}, z)), \alpha_{m(p)-1}, \alpha_{n(p)-1} \} \end{aligned} \tag{5}$$

Since $\{\alpha_n\} \rightarrow 0$ as $n \rightarrow \infty$, there exists a $P \in \mathbb{N}$ such that $\alpha_n < \varphi(\varepsilon)$ for all $n > P$. Hence by (5), for $n(p), m(p) > P+1$,

$$\varphi(\varepsilon) \leq K \varphi(\rho(x_{m(p)-1}, x_{n(p)-1}, z))$$

$$\leq K[\varphi(\rho(x_{m(p)-1}, x_{m(p)}, z)) + \rho(x_{m(p)}, x_{n(p)-1}, z) + \rho(x_{m(p)-1}, x_{n(p)-1}, x_{m(p)})]$$

$$\leq K\varphi(\rho(x_{m(p)-1}, x_{m(p)}, z)) + \varepsilon$$

Taking $P \rightarrow \infty$ and by using (4), we obtain

$$\varphi(\varepsilon) \leq K \varphi(\varepsilon) < \varphi(\varepsilon), \text{ which is a contradiction.}$$

Therefore $\{x_n\}$ is a Cauchy sequence. As X is complete, $\{x_n\}$ converges to x say in X.

Now consider

$$\begin{aligned} \varphi(\rho(Tx, x_n, z)) &\leq \varphi(\rho(Tx, Tx_{n-1}, z)) \\ &\leq K \max\{ \varphi(\rho(x, x_{n-1}, z)), \rho(x, Tx, z), \alpha_{n-1} \} \end{aligned}$$

Taking limits as $n \rightarrow \infty$, we have

$$\varphi(\rho(Tx, x, z)) \leq K\varphi(\rho(x, Tx, z))$$

$$\Rightarrow \varphi(\rho(Tx, x, z)) = 0 \Rightarrow Tx = x$$

Uniqueness of the fixed point follows evidently from (1). Hence the theorem.

1. Fixed point theorem for a pair of self maps

Theorem 2. Let (X, ρ) be a bounded complete 2-metric space and S and T be self maps of X such that $ST = TS$. Further, assume that S and T satisfy the following inequality:

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there exists $K \in (0, 1)$ and $\phi \in \Phi$ such that, for $z \in X$

$$\phi(\rho(Sx, Ty, z)) \leq K \max \left\{ \phi(\rho(x, y, z)), \phi(\rho(x, Sx, z)) + \frac{\phi(\rho(y, Ty, z))}{2} \right\} \quad (6)$$

for all $x, y, z \in X$. Then one of S and T (and hence both) have a unique common fixed point in X . For any $x_0 \in X$, we define the sequence $\{x_n\}$ in X by $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ for $n = 0, 1, 2, \dots$ (7)

To prove the theorem we need the following lemmas.

Lemma 1. Under the hypothesis of Theorem 2, for any $x_0 \in X$, the sequence $\{x_n\}$ defined by [7] satisfies the following inequalities:

$$\phi(\rho(x_{2n}, Tx_{2n}, z)) \leq K^{2n} \phi(\rho(x_0, Tx_0, z)) \quad (8)$$

$$\begin{aligned} \phi(\rho(x_{2n+2}, Tx_{2n+2}, z)) &\leq K \phi(\rho(x_{2n}, Tx_{2n}, z)) \\ &\leq K^{2n+1} \phi(\rho(x_0, Tx_0, z)) \end{aligned} \quad (9)$$

$$\phi(\rho(x_{2n+1}, Sx_{2n+1}, z)) \leq K^{2n} \phi(\rho(x_1, Tx_1, z)) \quad (10)$$

$$\begin{aligned} \phi(\rho(x_{2n+3}, Sx_{2n+3}, z)) &\leq K \phi(\rho(x_{2n+1}, Sx_{2n+1}, z)) \\ &\leq K^{2n+1} \phi(\rho(x_1, Sx_1, z)) \end{aligned} \quad (11)$$

Proof of (8):

$$\begin{aligned} \phi(\rho(x_2, Tx_2, z)) &= \phi(\rho(STx_0, Tx_2, z)) \\ &\leq K \max \{ \phi(\rho(Tx_0, x_2, z)), \phi(\rho(Tx_0, x_2, z)), \phi(\rho(x_2, Tx_2, z)) \} \\ &= K \phi(\rho(Tx_0, x_2, z)) \end{aligned} \quad (12)$$

Consider

$$\begin{aligned} \phi(\rho(Tx_0, x_2, z)) &= \phi(\rho(Tx_0, STx_0, z)), \\ &\leq K \max \{ \phi(\rho(x_0, Tx_0, z)), \phi(\rho(x_0, Tx_0, z)), \phi(\rho(Tx_0, x_2, z)) \} \\ &= K \phi(\rho(x_0, Tx_0, z)) \end{aligned} \quad (13)$$

By (12) and (13), we obtain

$$\phi(\rho(x_2, Tx_2, z)) \leq K^2 \phi(\rho(x_0, Tx_0, z))$$

Hence the inequality (8) is valid for $n = 1$. We shall assume that (8) is true for $n = p$ for some $p \in \mathbb{N}$, $p > 1$

$$\text{i.e., } \phi(\rho(x_{2p}, Tx_{2p}, z)) \leq K^{2p} \phi(\rho(x_0, Tx_0, z)) \quad (14)$$

Now consider .,

$$\begin{aligned} \phi(\rho(x_{2(p+1)}, Tx_{2(p+1)}, z)) &= \phi(\rho(STx_{2p}, Tx_{2(p+1)}, z)) \\ &\leq K \max \{ \phi(\rho(Tx_{2p}, x_{2(p+1)}, z)), \phi(\rho(Tx_{2p}, Tx_{2(p+1)}, z)), \\ &\quad \phi(\rho(x_{2(p+1)}, Tx_{2(p+1)}, z)) \} \\ &= K \phi(\rho(Tx_{2p}, x_{2(p+1)}, z)) \end{aligned} \quad (15)$$

and

$$\begin{aligned} \phi(\rho(Tx_{2p}, x_{2(p+1)}, z)) &= \phi(\rho(Tx_{2p}, STx_{2p}, z)) \\ &\leq K \max \{ \phi(\rho(x_{2p}, Tx_{2p}, z)), \phi(\rho(x_{2p}, Tx_{2p}, z)), \\ &\quad \phi(\rho(Tx_{2p}, STx_{2p}, z)) \} \end{aligned}$$

$$= K\varphi(\rho(x_{2p}, Tx_{2p}, z)) \tag{16}$$

By (15) and (16) and by using the induction hypothesis (14), we have

$$\varphi(\rho(x_{2(p+1)}, Tx_{2(p+1)}, z)) \leq K^2\varphi(\rho(x_{2p}, Tx_{2p}, z)) \leq K^{2(p+1)}\varphi(\rho(x_0, Tx_0, z))$$

Thus inequality (8) holds for $n = p + 1$. This completes proof of (8).

Proof of (9)

$$\begin{aligned} \varphi(\rho(x_{2n+2}, Tx_{2n+2}, z)) &= \varphi(\rho(STx_{2n}, Tx_{2n}, z)) \\ &\leq K \max \{ \varphi(\rho(Tx_{2n}, x_{2n}, z)), \varphi(\rho(x_{2n}, Tx_{2n}, z)), \\ &\quad \varphi(\rho(Tx_{2n}, x_{2n+2}, z)) \} \\ &= K\varphi(\rho(x_{2n}, Tx_{2n}, z)) \\ &= K^{2n+1}\varphi(\rho(x_0, Tx_0, z)) \text{ from (8).} \end{aligned}$$

This proves (9).

Proofs (10) and (11) are similar to (8) and (9) respectively.

Lemma 2. Under the hypothesis of theorem 2 assume that $x_n \neq x_{n+1}$ ($n = 0, 1, 2, 3 \dots$). Then for any $m, n \in \mathbb{N}$ with $n > m$, the following inequalities hold.

- i) $\varphi(\rho(x_{2n}, x_{2m}, z)) \leq K^{2m} \varphi(K)$
 - ii) $\varphi(\rho(x_{2n+1}, x_{2m+1}, z)) \leq K^{2m+1} \varphi(K)$
 - iii) $\varphi(\rho(x_{2n+1}, x_{2m}, z)) \leq K^{2m} \varphi(K)$
 - iv) $\varphi(\rho(x_{2n}, x_{2m+1}, z)) \leq K^{2m+1} \varphi(K)$
- where K is a diameter of X .

Proof.

Let $\beta_n = \rho(x_n, x_{n+1}, z)$ and $\alpha_n = \varphi(\beta_n)$, $z \in X$. By using the inequality (6), it can be easily seen that $\alpha_n \leq K\alpha_{n-1}$ ($n = 1, 2, 3 \dots$). (17)

We shall prove (i) – (iv) by induction on m .

$$\begin{aligned} \text{(i)} \quad \varphi(\rho(x_{2n}, x_2, z)) &= \varphi(\rho(Tx_{2n-1}, STx_0, z)) \\ &\leq K \max \{ \varphi(\rho(x_{2n-1}, Tx_0, z)), \varphi(\rho(x_{2n-1}, Tx_{2n-1}, z)), \\ &\quad \varphi(\rho(Tx_0, x_2, z)) \} \tag{18} \end{aligned}$$

$$\begin{aligned} \varphi(\rho(x_{2n-1}, x_1, z)) &= \varphi(\rho(Sx_{2n-1}, Tx_0, z)) \\ &\leq K \max \{ \varphi(\rho(x_{2n-2}, x_0, z)), \varphi(\rho(x_{2n-2}, Tx_{2n-2}, z)), \\ &\quad \varphi(\rho(x_0, Tx_0, z)) \} \tag{19} \end{aligned}$$

Put $n = 0$ in 9 of Lemma

1, we obtain

$$\varphi(\rho(x_2, Tx_0, z)) \leq K \varphi(\rho(x_0, Tx_0, z)) \tag{20}$$

Hence from (17), (18), (19) and (20) we have

$$\begin{aligned} \varphi(\rho(x_{2n}, x_2, z)) &\leq K^2 \{ \varphi(\rho(x_{2n-2}, x_0, z)), \varphi(\rho(x_{2n-2}, Tx_{2n-2}, z)), \\ &\quad \varphi(\rho(x_0, Tx_0, z)) \} \\ &\leq K^2\varphi(K) \end{aligned}$$

Hence (i) is true for $m = 1$.

Assume (i) is true for $m = 1$

$$\varphi(\rho(x_{2n}, x_{2(m+1)}, z)) = K^{2^{(m-1)}}\varphi(K) \quad n > m - 1 \tag{21}$$

Now consider

$$\begin{aligned} \varphi(\rho(x_{2n}, x_{2m}, z)) &= \varphi(\rho(Tx_{2n-1}, STx_{2m-2}, z)) \\ &\leq K \max \{ \varphi(\rho(x_{2n-1}, Tx_{2m-2}, z)), \varphi(\rho(x_{2n-1}, Tx_{2n-1}, z)), \\ &\quad \varphi(\rho(Tx_{2m-2}, x_{2m}, z)) \} \end{aligned} \tag{22}$$

Then we have

$$\begin{aligned} \varphi(\rho(x_{2n-1}, Tx_{2m-2}, z)) &= \varphi(\rho(Sx_{2n-2}, Tx_{2m-2}, z)) \\ &\leq K \max \{ \varphi(\rho(x_{2n-2}, x_{2m-2}, z)), \alpha_{2n-2}, \\ \varphi(\rho(x_{2m-2}, Tx_{2m-2}, z)) \} \end{aligned} \tag{23}$$

Put $n = m - 1$ in (9) of Lemma 1. Then we have

$$\varphi(\rho(x_{2n-2}, Tx_{2m-2}, z)) \leq K\varphi(\rho(x_{2m-2}, Tx_{2m-2}, z)) \tag{24}$$

Hence by (22), (23) and (24), we have

$$\varphi(\rho(x_{2n}, x_{2m}, z)) \leq K^2 \max \{ \varphi(\rho(x_{2n-2}, x_{2m-2}, z)), \alpha_{2n-2}, \varphi(\rho(x_{2m-2}, Tx_{2m-2}, z)) \} \tag{25}$$

By induction hypothesis (21),

$$\varphi(\rho(x_{2n-2}, x_{2m-2}, z)) \leq K^{2^{(m-2)}}\varphi(K) \tag{26}$$

Again by (9) of Lemma 1

$$\varphi(\rho(x_{2m-2}, Tx_{2m-2}, z)) \leq K^{2^{(m-1)}}\varphi(\rho(x_0, Tx_0, z)) = K^{2^{m-2}}\varphi(K) \tag{27}$$

By (25), (26) and (27), we obtain

$$\begin{aligned} \varphi(\rho(x_{2n}, x_{2m}, z)) &\leq K^2 \max \{ K^{2^{(m-1)}}\varphi(K), K^{2^{(n-1)}}\alpha_0, K^{2^{(m-1)}}\varphi(K) \} = K^{2^m} \max \{ \\ \varphi(K), K^{2^{(n-m)}}\alpha_0, \varphi(K) \} \\ &= K^{2^m}\varphi(K) \text{ which proves (i)} \end{aligned}$$

Similarly, we can prove (ii), (iii) and (iv) by using Lemma 1.

Thus we have the following result.

Proposition 1. Under the hypothesis of Theorem 2, for any $x_0 \in X$ in the sequence $\{x_n\}$ defined by $x_{2n+1} = Sx_{2n}$ and $x_{2n+2} = Tx_{2n+1}$ ($n = 0, 1, 2, \dots$) if $x_n \neq x_{n+1}$ for $n = 0, 1, 2, \dots$, then

$$\varphi(\rho(x_m, x_n, z)) \leq K^m \varphi(K) \quad (z \in X),$$

for any $m, n \in \mathbb{N}$ provided $n > m$, where K is the diameter of X .

By proposition we prove the Theorem 2.

Proof of Theorem 2: Let $x_0 \in X$. Let $\{x_n\}$ be defined by

$$x_{2n+1} = Sx_{2n} \text{ and } x_{2n+2} = Tx_{2n+1}, \quad (n \in \mathbb{N}).$$

Suppose for some $n \in \mathbb{N}$, $x_{2n+2} = x_{2n}$, then $Sx_{2n} = x_{2n+1} = x_{2n}$ such that x_{2n} is a fixed point of S .

Hence

$$\begin{aligned} \varphi(\rho(Tx_{2n}, x_{2n}, z)) &= \varphi(\rho(Tx_{2n}, Sx_{2n}, z)) \\ &\leq K \max \{ \varphi(\rho(x_{2n}, x_{2n}, z)), \varphi(\rho(x_{2n}, Tx_{2n}, z)), \\ &\quad \varphi(\rho(x_{2n}, Sx_{2n}, z)) \} \\ &= K \varphi(\rho(x_{2n}, Tx_{2n}, z)) \\ \Rightarrow \rho(Tx_{2n}, x_{2n}, z) &= 0 \Rightarrow Tx_{2n} = x_{2n} \end{aligned}$$

Uniqueness of common fixed point evidently follows from the inequality (6).

Similarly if $x_{2n+2} = x_{2n+1}$, then $Tx_{2n+1} = x_{2n+2} = x_{2n+1}$

$\therefore x_{2n+1}$ is a fixed point for T . Hence by (6) x_{2n+1} is also a unique fixed point for S .

We shall assume that $x_{2n} \neq x_{n+1}$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Then choose M such that $K^M \varphi(K) < \varphi(\varepsilon)$. Then for $n > m > M$, $z \in X$,

$$\varphi(\rho(x_m, x_n, z)) \leq K^m \varphi(K) < K^M \varphi(K) < \varphi(\varepsilon) \quad (\because \text{by Proposition 1})$$

$\Rightarrow \rho(x_m, x_n, z) < \varepsilon \Rightarrow \{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $z \in X$ such that $\lim_{n \rightarrow \infty} x_n = z$.

$$n \rightarrow \infty$$

We shall prove that $Tz = z$ and $Sz = z$

If possible, $Tz \neq z$. Then, for $u \in X$,

$$\begin{aligned} \varphi(\rho(Tz, x_{2n+1}, u)) &= \varphi(\rho(Tz, Sx_{2n}, u)) \\ &\leq K \max \{ \varphi(\rho(z, x_{2n}, u)), \varphi(\rho(z, Tz, u)), \varphi(\rho(x_{2n}, x_{2n+1}, u)) \} \end{aligned}$$

Taking limit as $n \rightarrow \infty$, as φ is continuous.

$$\varphi(\rho(Tz, z, u)) \leq K \varphi(\rho(z, Tz, u)) = K \varphi(\rho(Tz, z, u)) < \varphi(\rho(Tz, z, u))$$

which is a contradiction. Hence $\varphi(\rho(Tz, z, u)) = 0 \Rightarrow Tz = z$. This z is also a fixed point of S , by (6).

Hence the theorem.

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